GROUPS WITH $S_{2'}$-SUBGROUPS AND $S_{3'}$-SUBGROUPS†

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ABSTRACT

In this paper we formulate criteria for the solvability of groups of type $E_{2'}$ and $E_{3'}$

1. Introduction

All groups considered here are finite. Philip Hall's characterization of solvable groups asserts that the existence of $G_{p'}$ in a finite group G for all p implies that G is solvable.

Generally the existence of $G_{p'}$ in a finite group G for every $p \in \pi(G) - \{r\}$, where $r \in \pi(G)$ is prime, does not imply that G is solvable. For example, $G = \mathrm{PSL}(2,7) \times G_{p_1} \times \cdots \times G_{p_k}$, where $\pi(G) = \{2,3,7,p_1,\cdots,p_k\}$, satisfies $E_{p'}$ for every $p \in \pi(G) - \{3\}$, but G is not solvable. However, we conjecture that the existence of $G_{2'}$ and $G_{3'}$ in a group G forces G to be solvable. We cannot prove this conjecture, but we can state the following theorem.

THEOREM 1. *Let G satisfy $E_{2'}$ and $E_{3'}$. Assume also that G satisfies $E_{\{2,3\}}$. Then G is solvable.*

[4, Th. 18.7] implies that the above-mentioned Hall theorem is derived from Theorem 1.

Define $d(G)$ to be the maximum of the orders of the abelian subgroups of G. Let us denote the class of all groups satisfying $E_{2'}$ and $E_{3'}$ by θ.

THEOREM 2. *Let $G \in \theta$ be of order $|G| = 2^m 3^n p_1^{e_1} \cdots p_k^{e_k}$, where $\pi(G) = \{2,3,p_1,\cdots,p_k\}$. Assume that G satisfies one of the following:*

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(i) $d(G_3) < \prod_{p \in \pi} |G_p|$, *where* $\pi = \{p_i/p_i > n\}$.

(ii) $d(G_3) < d(G_{p_i})$ *for some* i.

(iii) $|G_3| < |G_{p_i}|$ *for some* i.

Then G *is not a nonabelian simple group.*

COROLLARY 3. *Let* $G \in \theta$ *and* $|G| = 2^m 3^n p_1^{e_1} \cdots p_k^{e_k}$. *If* $|\pi(G)| \geqq n$, *then* G *is not a nonabelian simple group.*

We need the following theorem on solvable groups.

THEOREM 4. *Let* G *be a solvable group of order* $|G| = p_1^{e_1} \cdots p_k^{e_k}$. *Define* $\pi = \{p_i/p_i > e_1,\ i \neq 1\}$. *Then* $p_1^{e_1} < |G_\pi|$ *implies that* $O_{p_1'}(G) \neq 1$.

In particular, if $|G| = q^\beta p^\alpha, q^\beta < p^\alpha$ *and* $p > \beta$ *then* $O_p(G) \neq 1$.

NOTE. Burnside [3] proved that $O_p(G) \neq 1$ if q and p are odd even for $p \leqq \beta$.

As a corollary of Theorem 4 we obtain the following.

COROLLARY 5. *Let* $G \in \theta$ *and* $|G| = 2^m 3^n p_1^{e_1} \cdots p_k^{e_k}$. *If* $|\pi(G)| \geqq m$, *then* G *is not a nonabelian simple group.*

All groups in this paper are assumed to be finite. Our notation is standard and taken mainly from [7]. In particular, $\pi(G)$ will denote the set of primes dividing $|G|$. The set of primes not in π, will be denoted by π'.

In accordance with Hall, we consider the following statements about a group G.

E_π: *G has an* S_π*-subgroup.*

C_π: *G has an* S_π*-subgroup and any two such subgroups are conjugate.*

D_π: *G satisfies* C_π *and every* π*-subgroup of G is contained in an* S_π*-subgroup.*

Finally, if G satisfies E_π, then G_π will denote an S_π-subgroup of G.

2. Preliminary results

The proof of Theorem 1 depends upon the following lemma.

LEMMA 6. *If* $G \in \theta$ *then:*

(i) *if* $2 \nmid |G|$ *or* $3 \nmid |G|$ *then* G *is solvable.*

(ii) *if* $2,3 / |G|$ *then* $G_{2'}$ *and* $G_{3'}$ *are solvable.*

PROOF. (i) If $2 \nmid |G|$ then G is solvable by [6]. If $3 \nmid |G|$ let G be a minimal counterexample. If N is a proper nontrivial normal subgroup of G, then by induction N is solvable. Similarly $G|N$ is also solvable. Therefore G is solvable, a contradiction. We conclude that G is simple, and assuming Thompson's classification of simple 3′-groups we find that G is isomorphic to $S_z(q)$. But $S_z(q)$ does not have $G_{2'}$, a contradiction. Hence G is solvable.

(ii) $G_{2'}$ is solvable by [6]. $G_{3'}$ is a θ-group by [4, Th. 18.6], and hence it is solvable by (i).

We need also the next lemma.

LEMMA 7. *If $G \in \theta$ is a simple group, $K = G_{3'}$ or $K = G_{\{2,p\}}$ a prime, and $R = G_{2'}$ or $R = G_{\{3,q\}}$, q a prime, then $O_{2'}(K) = O_{3'}(R) = 1$.*

PROOF. Assume that G is a nonabelian simple group. Lemma 6(ii) implies that $G_{2'}$ and $G_{3'}$ are solvable. Let K be $G_{3'}$. If $O_{2'}(K) \neq 1$ then there exists a prime $p \neq 2,3$ such that $O_p(K) \neq 1$. Clearly $O_p(K) \subseteq G_{2'}^x$ for some x in G. Since $G_{2'}^x$ has the same properties as $G_{2'}$, we can replace $G_{2'}$ by $G_{2'}^x$ and so can assume without loss that $G_{2'}$ contains $O_p(K)$. Now by [7, Lem. 6.4.2], we have $G = G_{2'}K$. Hence if $g \in G$, then $g = k \cdot r$ where $k \in K$ and $r \in G_{2'}$. Therefore $O_p(K)^g = O_p(K)^r \subseteq G_{2'}$. But then $1 \subset O_p(K) \subseteq \bigcap_{g \in G} G_{2'}^g \lhd G$. Hence G is not simple, a contradiction. Therefore $O_{2'}(K) = 1$.

Similarly $O_{3'}(G_{2'}) = 1$. If $K = G_{\{2,p\}}$ or $R = G_{\{3,q\}}$ we use the same method.

LEMMA 8. *Let K be a finite subgroup of $GL(n, \overline{GF(t)})$, where t is a prime. Assume that $(|K|, t) = 1$ and if $p \in \pi(K)$ then $p > n$. Then K is abelian.*

PROOF. We prove this by induction on n. If $n = 1$ then K is a cyclic subgroup. By assumption char $\overline{GF(t)} \nmid |K|$, hence K is completely reducible. Suppose that K is reducible $K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$.

By induction, K_i (where $i = 1$ or 2) is an abelian subgroup of K, as $|K| = |K_1| \cdot |K_2|$. Hence K is abelian. It remains to deal with the case where K is an irreducible $\overline{GF(t)}$-representation. [12, Th. 15.11 and Th. 15.10] now imply that $n / |K|$. But by assumption $n \nmid |K|$, a contradiction. Therefore K is abelian.

As an immediate corollary of Lemma 8 we have Lemma 9.

LEMMA 9. *Let K be a subgroup of $GL(n,3)$, $n \geqq 3$, such that if $p \in \pi(K)$ then $p > n$. Then K is abelian.*

LEMMA 10. *Let $G = G_\pi G_{\pi'}$. Assume that $G_\pi \neq 1$ is abelian and that $|G_\pi| > |G_{\pi'}|$. Then $O_\pi(G) \neq 1$.*

PROOF. By assumption $G_\pi \cap G_\pi^x \neq 1$ for all $x \in G$. Since G_π is abelian, Herzog in [9, Th. 1] implies that there exists $x \in G$ such that $O_\pi(G) = G_\pi \cap G_\pi^x \neq 1$.

As an immediate corollary of Lemma 10 we obtain Lemma 11.

LEMMA 11. *If K is an abelian subgroup of $GL(n,q)$, and if $(|K|,q) = 1$, then $|K| < q^n$.*

PROOF. Let R be the group of order q^n such that $K \underset{\sim}{\subseteq} \mathrm{Aut}(R)$. Embed K and R in the natural manner in their semi-direct product KR. Clearly $R \lhd KR$. If $|K| > q^n$ then Lemma 10 implies that there exists a nontrivial element $k \in K$ such that $k \in O_{\pi(K)}(KR)$. Hence $k \in C_{KR}(R)$, a contradiction. Therefore $|K| < q^n$.

We conclude with the trivial lemma.

LEMMA 12. *Let G be of order $|G| = 2^m 3^n p_1^{e_1} \cdots p_k^{e_k}$. Define $\pi_m = \{p_i/p_i > m\}$ and $\pi_n = \{p_i/p_i > n\}$. Assume that (i) $k \geqq n-2$ and $n \geqq 6$ or (ii) $k \geqq m-2$ and $m \geqq 4$. Then $|G_{n_n}| > 3^n$ or $G_{\pi_m}| > 2^m$, respectively.*

The proof is by induction on n and m.

3. Some properties of groups belonging to θ

For any group G let $m(G)$ denote the minimal number of generators of G. Define $m_s(G) = \max m(U)$ where U ranges over all the subgroups of G. Define $m_A(G) = \max m(A)$ where A ranges over all the abelian normal subgroups of G. If P is a p-group, $m_s(P) \leqq n$, and K is a p'-subgroup of $\mathrm{Aut}(P)$, it is known that $K \underset{\sim}{\subseteq} GL(n,p)$.

We need the following preliminary result.

PROPOSITION 13. *Let $G \in \theta$ be a simple group. Set $m_2 = m(O_2(G_{3'}))$ and set $m_3 = m(O_3(G_{2'}))$. Assume that $p = 2$ or 3 and that*

$$(|G|, p^s(p^{m_p} - 1)(p^{m_p - 1} - 1) \cdot \cdots \cdot (p-1)) = 2^\alpha 3^\beta r^\gamma$$

where s, α, β and γ are non negative integers and r is a prime. If $r = 5, 7, 13$ or 17, assume also that G_r is cyclic. Then G is simple of prime order.

PROOF. Assume that G is a nonabelian simple group. Lemma 7 implies then that $O_{2'}(G_{3'}) = O_{3'}(G_{2'}) = 1$.

Since $G_{2'}$ and $G_{3'}$ are solvable (Lemma 6), Hall-Higman [Lem. 1.2.3] implies that $C_{G_{3'}}(O_2(G_{3'})) = Z(O_2(G_{3'}))$ and that $C_{G_{2'}}(O_3(G_{2'})) = Z(O_3(G_{2'}))$. Set $N = N_{G_{3'}}/(O_2(G_{3'}))$ and set $C = C_{G_{3'}}(O_2(G_{3'}))$. Then $N/C = G_{3'}/Z(O_2(G_{3'}))$ is isomorphic to a subgroup of $\mathrm{Aut}(O_2(G_{3'}))$. [13, Th. 7.3.11] implies that $|N/C| / |\mathrm{Aut}(O_2(G_{3'}))| / 2^k \prod_{0 \leqq j \leqq m_2 - 1} (2^{m_2 - j} - 1)$, where k is an integer. By assumption $|N| = |G_{3'}| = 2^\alpha r^\gamma$. Therefore $|G| = 2^\alpha 3^\beta r^\gamma$. Burnside's $p^a q^b$ theorem yields that $\alpha, \beta, \gamma > 1$. Thompson's theorem on minimal simple groups implies that r must be either 5, 7, 13 or 17. Hence, by assumption, G_r is cyclic. From Leon's theorem [11] and the recent results of Leon and Wales we obtain that G is one of the following: $L_2(q)$ with $q = 5, 7, 8, 9$ or 17, $L_3(3)$, $U_4(2)$ or $U_3(3)$. In every case G does not have both $G_{2'}$ and $G_{3'}$, a contradiction. Therefore G is simple of prime order. The second part of this proposition for $p = 3$, is proved similarly.

PROPOSITION 14. *Let $G \in \theta$ satisfy one of the following:*

(i) *$G_{2'}$ is 3-nilpotent.*

(ii) *$G_{3'}$ is 2-nilpotent.*

(iii) *G_2 is abelian.*

Then G is solvable.

PROOF. Let G be a minimal counterexample. (i) Let N be a proper nontrivial normal subgroup of G. Then by induction N and G/N are solvable, a contradiction. Hence G is simple. Lemma 7 then implies that $O_{3'}(G_{2'}) = 1$. As $G_{2'}$ is 3-nilpotent, $G_{2'}$ is a 3-group and G is solvable by Burnside's theorem, a contradiction. (ii) The proof of this part is similar. (iii) Clearly, by induction, G is simple. By Walter's theorem [14], [15], and [16] we obtain that $G \notin \theta$, a contradiction. Therefore G is solvable.

PROPOSITION 15. *Let $G \in \theta$ satisfy one of the following:*

(i) $m_s(G_2) \leqq 4$.

(ii) $m_s(G_3) \leqq 4$.

(iii) $m_A(G_3) \leqq 2$.

Then G is solvable.

In particular if $|G_2| \leqq 2^5$ or $|G_3| \leqq 3^4$ and $G \in \theta$, then G is solvable.

PROOF. Let G be a minimal counterexample.

(i) By induction G is a non-abelian simple group. From Lemma 7 and Hall-Higman [Lem. 1.2.3] we obtain:

$$|N_{G_{3'}}(O_2(G_{3'}))/C_{G3'}(O_2(G_{3'}))| = |G_{3'}/Z(O_2(G_{3'}))|\,2^k\,|GL(4,2)|$$

for some k. Therefore one of the following holds:

(1) $|G_{3'}| = 1$.

(2) $|G_{3'}| = 2^\alpha$.

(3) $|G_{3'}| = 2^\alpha \cdot 7$.

(4) $|G_{3'}| = 2^\alpha \cdot 5$,

(5) $|G_{3'}| = 2^\alpha \cdot 5 \cdot 7$.

Clearly $\alpha > 2$, by Proposition 14 (iii). By Proposition 13 we have only case (5). Since $G_{3'}$ is solvable, G has an $S_{\{5,7\}}$-subgroup $G_{\{5,7\}}$. Since

$$G_{\{5,7\}}Z(O_2(G_{3'}))/Z(O_2(G_{3'})) \underset{\sim}{\subseteq} \mathrm{Aut}(O_2(G_{3'}))$$

we obtain that $G_{\{5,7\}} \underset{\sim}{\subseteq} GL(4,2)$. But Satz [27, (7.3), Chap. II] implies that $GL(4,2)$ does not contain subgroups of order $5,7$, a contradiction, and (i) is proved.

(ii) By induction, G is simple, and $m_s(G_3) \leqq 4$. In particular $m = m(O_3(G_{2'})) \leqq 4$. As in the proof of (i), it follows that $G_{2'}/Z(O_3(G_{2'}))$ is isomorphic to a subgroup of $\mathrm{Aut}(O_3(G_{2'}))$ and its order divides $3^k|GL(m,3)|$ for some k. Therefore one of the following holds:

(1) $|G_{2'}| = 1$.

(2) $|G_{2'}| = 3^\alpha$.

(3) $|G_{2'}| = 3^\alpha \cdot 13$.

(4) $|G_{2'}| = 3^\alpha \cdot 5$.

(5) $|G_{2'}| = 3^\alpha \cdot 5 \cdot 13$.

Since $G_{3'}$ is solvable, we obtain in Case 5 that

$$G_{\{5,13\}} \underset{\sim}{\subseteq} GL(4,3).$$

By Proposition 13 we have that $|G_{2'}| = 3^\alpha \cdot 5 \cdot 13$. But Satz [10, (7.3), Chap. II] implies that $GL(4,3)$ does not satisfy $E_{\{5,13\}}$, a contradiction. Therefore G is solvable.

(iii) If $m_A(G_3) \leqq 2$ then Thompson's theorem (Satz [10, (12.3), Chap. III]) implies that $m_s(G_3) \leqq 3$. Therefore G is solvable by part (ii).

If $|G_2| \leqq 2^5$ then G_2 is abelian or $m_s(G_2) \leqq 4$. Proposition 15(i) and Proposition 14(iii) then imply that G is solvable.

4. Proofs of the theorems

Proof of Theorem 1. Let G be a minimal counterexample. Lemma 6 implies that $2,3/|G|$. Burnside's p^aq^b theorem implies that $|\pi(G)| \geqq 3$. Lemma 6 implies that $G_{2'}$ and $G_{3'}$ are solvable. Burnside's p^aq^b theorem implies that $G_{\{2,3\}}$ is solvable. Hence [7, Th. 6.4.4] implies that G is solvable.

Proof of Theorem 2. (i) If G is a nonabelian simple group, then Lemma 7 implies that $O_{3'}(G_{2'}) = 1$. From Hall-Higman [Lem. 1.2.3] we obtain that $N_{G_{2'}}(O_3(G_{2'}))/C_{G_{2'}}(O_3(G_{2'})) = G_{2'}/Z(O_3(G_{2'})) \underset{\sim}{\subseteq} \mathrm{Aut}(O_3(G_{2'}))$. Now Lemma 9 implies that $G_\pi \cdot Z(O_3(G_{2'}))/Z(O_3(G_{2'}) \underset{\sim}{\subseteq} GL(n,3)$ is an abelian subgroup if $n \geqq 3$ Therefore G_π is an abelian subgroup if $n \geqq 3$. By definition, $O_2(G_{2'}) = 1$ and $G_{2'}$

is of odd order. Let A be an abelian subgroup of $G_{2'}$ of maximal order. By assumption $d(G_3) < |G_\pi|$, therefore if $n \geqq 3$ then there exists $p \in \pi$, p a prime, such that $p/|A|$ and $p \geqq 5$. [1, Th. 1] then imlpies that $1 \subset O_p(A) \subseteq O_p(G_{2'}) = 1$ a contradiction. Hence $n \leqq 2$. Proposition 15 then implies that G is not a non-abelian simple group, proving (i).

(ii) Let A be an abelian subgroup of $G_{2'}$ of maximal order. By assumption $d(G_3) < d(G_p)$ for some $p > 3$. Therefore there exists $q \in \pi(G_{2'})$, $q \geqq 5$ a prime, such that $q/|A|$. [1, Th. 1] implies that $1 \neq O_q(A) \subseteq O_q(G_{2'})$. But $O_q(G_{2'}) = 1$ by Lemma 7, a contradiction, proving (ii).

(iii) Burnside's theorem [3] implies that $O_{p_i}(G_{\{3,p_i\}}) \neq 1$. Hence G is not a nonabelian simple group by Lemma 7.

Proof of Corollary 3. If $n \leqq 4$ then Proposition 15 implies that G is solvable. Define $\pi_n = \{p_i/p_i > n\}$. If $n \geqq 6$, Lemma 12 implies that $3^n < |G_{\pi_n}|$. Therefore $d(G_3) < |G_{\pi_n}|$. From Theorem 2(i) we obtain that G is not a nonabelian simple group. If $n = 5$ then $\pi(G) = \{2,3,5,11,13\}$ since $|G_{2'}/Z(O_3(G_{2'}))| / 3^k |GL(5,3)|$ and $\pi(GL(5,3)) = \{2,3,5,11,13\}$. As above $G_{\{11,13\}} \underset{\sim}{\subset} GL(5,3)$. But $GL(5,3)$ does not contain a $\{11,13\}$-subgroup, completing the proof.

Proof of Theorem 4. If $O_{p_1'}(G) = 1$, then $C_G(O_{p_1}(G)) = Z(O_{p_1}(G))$. Therefore $G/Z(O_{p_1}(G)) \underset{\sim}{\subset} \mathrm{Aut}(O_{p_1}(G))$. In particular, $G_\pi \underset{\sim}{\subset} GL(e_1, p_1)$. From Lemma 8 we obtain that G_π is abelian. Therefore $|G_\pi| < p_1^{e_1}$ by Lemma 11, a contradiction. Therefore $O_{p_1'}(G) \neq 1$.

Proof of Corollary 5. If $m \leqq 5$ then Proposition 15 implies that G is solvable. Hence $m \geqq 6$. Define $\pi_m = \{p_i/p_i > m\}$. Lemma 12 implies that $|G_{\pi_m}| > 2^m$. Lemma 6 and Theorem 4 imply that $O_{3'}(G_{2'}) \neq 1$. By Lemma 7, G is not a nonabelian simple group.

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